Selected contexts in the philosophy of mathematics

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Abstract
The present paper attempts to answer the question of whether it is reasonable to distinguish mathematical objects ontologically. It aims to demonstrate that mathematical objects can be divided into mathematical objects existing on their own and mathematical objects constructed by human beings. The present research also seeks to answer the questions raised by the philosopher Maco in his communication in response to my previous research. The paper presents the view that the ontological status of Lie groups and Lie algebras differs according to whether they are complex Lie groups and Lie algebras, respectively. Other questions were addressed on the status of Hilbert spaces, whether the set of natural numbers and the set of hyperreal numbers are independent. The present attempt to answer these questions does not, in my opinion, depart from the Wittgensteinian approach to mathematics, which is what I am trying to prove.
Key words: ontological status, mathematical object, Wittgenstein's philosophy of mathematics, Lie groups

Introduction

Attitudes towards mathematical objects and their ontological status vary across the mathematical domain. Not surprisingly, research has shown that attitudes of prospective mathematics lecturers towards mathematics differ considerably (Jancarík, Marcom, Kleinke, 2023). The present paper aims to critically analyse the thesis that mathematical objects can be divided into two groups from an ontological point of view. The first group are essentially ontological objects in the sense of their Platonic existence. I consider the other group of objects problematic from an ontological point of view. Since they do not have an established ontological existence, I make the conjecture that they are objects constructed by humans (Ambrozy, 2019). I do not doubt the utility of mathematical entities from either group; "the utility of mathematical objects is somewhat disconnected from their ontological status" (Freeman, 2022: 15). Duke argues that the debate between nominalists and those who favour the actual association of abstract singular terms with objects may end in a compromise (Duke, 2012). Some believe "mathematical objects retain their identity through different axiomatisations" (Christopoulou, 2019: 383). In Hilbert, for example, we may encounter implicit definitions of basic geometric entities - point, line, etc. (Tselishchev, 2013). Husserl presents the view that mathematical objects have an ideal nature. This is traced back to Plato. Insofar as Descartes refers to the eternal truths of mathematics, this is not Platonism but a view derived from Proclus (Hattab, 2016). There is also a structuralist view of the nature of mathematical objects - cf. (Pleitz, 2010). Popper seeks to reconcile the position between construction and discovery; in his view, this contradiction in mathematics is only illusory (Harada, 2005). Sucharek perceives the problem more complexly, he asks about the origin of the idea (Sucharek 2016). Molini distinguishes between strong and weak mathematical objectivity (Molini, 2020). Regarding mathematical Platonism, Baker seeks to postulate an Enhanced Indispensability Argument (EIA) to support it (Baker, 2009). Drekalovic considers this argument applicable only in an ideal mathematical framework, i.e., not the whole way the mathematical community produces real mathematical results (Drekalovic, 2022). The view presented in Ambrozy (2019) can be supported by the philosophy of mathematics of several philosophers. Of course, doubts and objections can be raised against the presented view. These are not objections postulated from other positions in the philosophy of mathematics, such as logicism and formalism. Some have been raised in correspondence discussions by the philosopher Róbert Maco. The latter gives some open questions that are not trivial and make the whole conception problematic.

The present paper builds on my paper (Ambrozy, 2019), in which I argue that the ontological status of mathematical objects is not naturalistic. I do not claim that the world of mathematical objects is located in the physical world. I argue that the world of mathematics is not subjective. In principle, it is possible to divide mathematical objects into two domains. The first group consists of mathematical objects existing independently of humans and the physical world. Man discovers and recognises them irrespective of the cultural environment in which he finds himself and his preunderstanding. The results of the top mathematics of the staff of the Academies of Sciences correspond to the results of the mathematicians of the natural nations, as Claude Levi-Strauss has convincingly demonstrated. Mark Steiner rejects Platonism in mathematics (Steiner, 2014). Platonism in mathematics is supported by many eminent mathematicians, for example, Gödel (Budiansky, 2021). His two incompleteness theorems do not undermine the ontological nature of mathematical objects. Thus, there are mathematical objects that exist objectively. In the same way, we can construct these mathematical objects. However, there are also mathematical objects that, although we can construct them mathematically, would not exist without human intervention. The examples include complex numbers and the δ-function, which J. von Neumann called the problematic entity. In conceiving the solution above, I also drew on discussions with the mathematician Lev Bukovský.

Maco's objections and questions in the field of the ontological nature of mathematical objects

Maco suggests that ontological differentiation is too bold in the issue under consideration and introduces unnecessary complications. He perceives a view that evaluates the apprehension of the issue from an ontological perspective as one that introduces a new conglomeration of issues rather than offering a path to a solution. Maco negatively evaluates the above perspective as hardly viable; he considers the division outlined unsustainable. His objections are condensed into a few points.

In the discussions, Maco argues that if the second group of objects in my conception contradicts the mathematical laws of a given field and is constructed as new (Ambrozy, 2019), in such a case, we need to include negative numbers in it as well. This is because they were initially understood as unnatural, as fictitious, because this contradicted the rules of what were then considered numbers. This would, therefore, mean that, for Maco, real numbers would also be in the realm of mathematical constructs. According to this philosopher, the ontological dualism in question would also give rise to other new questions to which it is difficult to formulate an answer. Can we regard Lie groups, Lie algebras, and Hilbert spaces as constructions, or do they apply to the first group of objectively existing mathematical objects? If the first group of independently existing objects were real numbers, what would be the ontological status of hyperreal numbers? In such a conception, is the set of natural numbers independent, or only its elements? I find Maco's counterarguments and questions relevant, and the task of the study will be to try to answer them in the context of whether they lead up to a falsification of the thesis of the division of mathematical objects into independently existing objects and human constructs.

Attempt to answer objections and questions

First, I will point out that in my paper, to which Maco responds in the correspondence discussion, constructed mathematical entities contradict mathematical laws in the field in many ways. I do not claim this in the absolute sense that they contradict them in all respects. As a first argument, negative numbers would also belong in the field if they violate the rules of what has been taken to be a number.

We consider it inadequate to proceed in such a way, assuming that the natural objectively existing mathematical object is only the current state of knowledge of mathematics. "In the opinion of D. E. Smith, an analysis of the languages of the Australian tribes showed that thirty of them had no numeral for numbers greater than four" (Folta, 2004: 43). Would this mean that, for example, the number 117, a concept apparently unknown to these tribes, should be considered, according to such an invariant, as a mathematical construct that has no support in existing mathematical objects? In this case, confusing a concrete number with an indeterminate concept only solves a little. The important factor of when we can speak of a transition from practical calculus to a theoretical discipline also plays a role.

It is well known that Chinese mathematics worked with negative numbers more than 2000 years ago. "Like Euclid, this is a compendium of the mathematical concepts and techniques which had been developed slowly from perhaps the Zhou (or Chou) dynasty (begins c.1000 BCE) through the Western Han dynasty" (Mumford, 2010: 120). In India, as early as the 4th century BCE, the Arthas¯astra, written by Kautilya, exists. It also discusses accounting. This mentions that accounts can show a deficit, and people can also have a negative net worth. Here, Kautilya still does not speak expresis verbis about negative figures. The Brâhma-sphuta-siddhânta, authored by Brahmagupta (7th century) already explicitly deals with the concept of negative number (Dutta, 2005). The Brahmagupta "describes how the basic operations work with zero" (Svitek, 2023: 173). Certainly, negative numbers can be seen as a natural mathematical object, for example, in connection with the notion of debt. In Europe, a certain notion of negative numbers can be found in Ptolemy (one case), but it is an implicit presence. In contrast, the notion of negative mathematical quantities has to be essentially evaporated (Mumford, 2010: 115-119). Meanwhile, Euclid did not use either the notion of zero or a negative number. An existing tradition since the 18th century B.C., which Ptolemy also reflected, Euclid circumvented the reduction of arithmetic and algebra to geometry.

Negative numbers are mentioned expressly by Al-Khwarizmi in Aljabr w'al muqabala, albeit in a single passage, which is the part dealing with multiplication. Negative numbers are also found in Leonardo of Pisa. In the course of solving the algorithm for solving linear equations in many unknowns, negative numbers appear in his work. They take the form of negative money owed. "Leonardo is making the first tentative steps towards enlarging the number system to include negatives" (Mumford, 2010: 130). The universal genius of the 14th century, little appreciated by history, Nicole Oresme, in his Tractatus de configurationibus qualitatum et motuum, did not reach the postulation of a negative number, a negative numerical value. However, he exceeded the limits set by Euclid. The work Summa de arithmetica, geometria, proportioni et proportionalita by Luca Pacioli (1445 - 1517) refers to negative numbers in the form of debt. The egg in the mathematical example given by Pacioli has a negative price. This is not an innovative mathematical treatise but a survey work intended to summarise arithmetic, geometry, and applied mathematics knowledge. Nota bene, it significantly impacted the development of double-entry bookkeeping. Even the eminent mathematician Girolamo Cardano, a 16th-century mathematician, admits negative solutions to some of his equations in his solution of 13 cases of the cubic equation and 44 derivative cases. In the chapter On the rule for postulating a negative in the book of the same title as Raymond Lull's well-known work Ars Magna, he calls them fictitious or negative. Similarly, he expresses a negative number in the form of a debt as a negative property. A few generations later, Galileo Galilei. This physicist clearly expressed the concept of the negative vertical velocity of a projectile at constant acceleration caused by gravity. Rather strikingly, negative numerical values were not introduced in their works by such eminent mathematicians as René Descartes and Pierre de Fermat.

It was not until the appearance of John Wallis and his Treatise on Algebra (1685) that European mathematics began to embrace negative numbers fully and systematically. Isaac Newton took a similar approach to the issue in his Philosophiæ Naturalis Principia Mathematica. "In Chapter 16, Addition, Subduction, Multiplication and Extraction of Roots in Specious Arithmetic, Wallis defines negative numbers as nicely, simply and clearly as you could wish" (Mumford, 2010: 137); (Vašek, Blaščíková & Nemec, 2022: 2-3). Thus, Wallis was the first mathematician in Europe to use the whole number series fully with negative numbers. However, as we have seen, he was not the first mathematician in Europe to use the notion of a negative number. Like Wallis, Newton also dealt with negative numbers. British mathematicians still had reservations about the notion of negative numbers 150 years after Wallis and Newton's appearance, even though in continental Europe, the notion was already part of the itinerary (Maz & Rico, 2007). Euclid and his approach seem to have hindered making negative numbers explicit in European mathematics for many centuries. Although there are differences in the way cultures have developed, how mathematics has progressed in them, and differences in its history, it has eventually become conceptually unified.

Cultural differences worldwide have caused a highly differentiated development of mathematics, reflected in the approach to negative numbers. While Chinese and Indian philosophy approached them without prejudice, in European mathematical culture, negative numbers were discussed only by selected mathematicians and often about debt as negative money. In the Arab world, this was done by the famous mathematician Al-Khwarizmi. In Europe, negative numbers became fully established in theoretical mathematics, mainly thanks to Wallis and Newton, yet British mathematicians rejected them for half a century afterward.

I believe referring to the fact that mathematics has rejected negative numbers for some time is not a compelling argument. Rejecting negative numbers was not a worldwide mathematical phenomenon; it only happened in Europe. Even there, mathematicians such as Leonardo of Pisa, Luca Pacioli, Girolamo Cardano, and the physicist Galileo Galilei, at least implicitly, admitted negative numbers. The reference to the fact that mathematics at a certain time rejected negative numbers, therefore, does not endure since this was not said in unison by eminent mathematicians around the world but only by mathematicians who paid tribute to European mathematics. It is, therefore, not a universal phenomenon but only a matter of the history of European mathematics.

Even within European mathematics, mathematicians such as Pacioli and Cardano appeared, who naturally perceived the negative number as a debt and deprivation. Therefore, in my view, it is not appropriate to refer to the fact that there was a period in one of the cultures during which there was a distrust of the notion of a negative number. Plato, in the dialogue Πολιτεία, clearly discusses the realm of mathematical objects that are above the χωρισμός. Operations on numbers generate negative numbers in a rather trivial way. Even without Plato's metaphysics, the simple notion of length, distance under the surface of a body of water, etc., can no longer conjure up the notion of a negative number by any difficult metaphysical abstraction. So, I do not see the point of moving it into a second category of man-constructed objects just because it was considered a fictional domain at certain periods in the development of European mathematics. It would be an unnecessary adaptation to a certain historical paradigm charged with the development of mathematics in a particular time and space. As I have shown above, such an approach, literally mesmerised by the historical situation, would relocate even the higher natural numbers into the man-constructed domain, which is unsustainable. Certainly, such an approach would, in Maco's view, also relocate the real numbers into the same domain. However, as we have shown, this is not the entire domain of world mathematics and is governed by certain developments in mathematics. Moreover, clinging to one paradigm and determining the ontological status of the objects of mathematics based on that paradigm is unjustifiable.

The other question Maco raises is whether the set of natural numbers is independent in this sense or just its elements. After all, the set of natural numbers is the set of all existing natural numbers. To conceive of these numbers as a set, that is, a particular set, is not some contorted metaphysical operation, and there is no need to create a new entity there. The ontological status of sets of mathematical objects is essentially a hidden question after the nature of universals, where one can take a position of moderate realism. Just as individual numbers are possible and real, there is no problem in perceiving them simultaneously as constituting a set. The set of numbers is merely the perception of them as a whole, not the creation of an originally non-existent mathematical object that would not have existed without human intervention. The elements of any mathematical entity do form a common whole, perceivable also as a set. Meanwhile, the set N_n is defined in the standard Fregean way through the direct successor relation S, i.e., logical means (Kolman, 2008: 440).

Next, Maco asks what ontological status Lie groups, Lie algebras, and Hilbert spaces would have. A Lie group is a kind of mathematical object that is simultaneously a group and a differentiable variety, in which case the group operations are compatible with a smooth structure on the variety. These are objects in which their two aspects, the algebraic (they are groups) and the geometric or differential topological (they are smooth varieties), live side by side in happy symbiosis (Fecko, 2004). Sophius Lie discovered this mathematical object in the context of the study of solutions of differential equations. Wilhelm Killing also discovered them. "Lie groups combine elements from several mathematical fields - analysis, algebra and geometry" (Pravda, 2007: 219). To understand a Lie group, one must first understand the concept of a group. "A group is a set G with mapping from $G \times G$ to G (denoted by $g * h$) that satisfies the following properties:

i) Associativity: for all $g, h, k \in G$, $g * (h * k) = (g * h) * k$.

(ii) There exists a unit element $e \in G$ such that for all $g \in G$, $g * e = e * g = = g$.

(iii) To every $g \in G$ there exists an inverse element $g\text{-}1 \in G$ for which $g\text{-}1 * g = g * g\text{-}1 = = e$" (Pravda 2007: 220). The group concept is based on associativity, the neutral element in operations, and the reciprocal number in every integer. It is the study of common properties, where similar pairs of sets and operations are sought, which form a group. A simpler variant of the introduction of the Lie group is the matrix Lie group. "A matrix Lie group G is any subgroup of the group GL (n, C) for which: let Am be any sequence of matrices of G. If Am converges to some matrix A of Mn(C), then $A \in G$ or A is singular" (Pravda, 2007: 221). Lie algebras show a connection with associative linear algebras. It is true that every matrix Lie group possesses a corresponding Lie algebra, and the transition from one structure to another simplifies the problems to be solved. It can be defined as follows: "Let G be a matrix Lie group. The Lie algebra g of this group is the set of all matrices X for which $e^{tX} \in G$ for all real t" (Pravda, 2007: 225). A Lie algebra can be associated to every Lie group. In general, the definition according to which a Lie group G is a smooth variety with group structure is valid, where the multiplication $\mu: G \times G \rightarrow G$ is a smooth mapping. Smooth varieties are essentially general spaces with sufficient Euclidean space properties for derivation and integration possibilities.

It is necessary to decompose the entities in question into individual components. A group as a mathematical entity is a certain arrangement of natural mathematical objects. The inverse element, the representation, the associativity, and the mathematical operations used here have no unnatural nature to be constructed. Lie groups are essentially the union of groups and smooth variety. "The union of the algebraic notion of the group and the differential-topological notion of (analytic) variety gives rise to a Lie group" (Fecko, 2004: 21). In mathematics, variety means a topological space locally similar to an n-dimensional Euclidean space; the vectors in question are defined on it. Usually, this term is understood to mean a smooth variety. The variety itself can be introduced through maps and atlases. If we have maps to Cn, we speak of a complex variety. A complex Lie group is a Lie group, which is also a complex variety, i.e., a variety with a holomorphic atlas (Šmíd, 2010: 1). Thus, unless a Lie group is a complex Lie group, it is not one of those mathematical objects we might consider pure constructions because it uses terms that do not contain components that contradict mathematical rules. The complex Lie group encompasses the domain of complex numbers; thus, it contains mathematical objects that exist as human constructs. It thus has the nature of an object that, as Maco would say, exists only for us.

"The representation of a group automatically induces also a certain (derived) representation of its Lie algebra, which is, in general, a homomorphism of the Lie algebra to the Lie algebra of (all) linear operators (in a fixed linear space)" (Fecko, 2004: 22). The notion of a Lie algebra is a simpler object than the group itself. A Lie algebra can be assigned to every Lie group (Zlatoš, 2011). As for the complex Lie algebra, these are bilinear forms on the complex vector space Cm. We can also associate a Lie algebra with a complex Lie group. The same is true for the Lie algebra as far as the complex Lie algebra is concerned.

The ontological status of Hilbert space may also raise questions. In this case, we will also proceed with subcomponent analysis. We can conclude that it is a composite mathematical object. A Hilbert space E is a vector space E such that all limits of arbitrary sequences of vectors from E on which a scalar product is defined. A vector space E is a set of E vectors with scalar multiplication and addition operations over complex or real numbers. Since the Hilbert space is also defined using complex numbers, which I consider to be constructions of mathematicians, it can be considered a complex object that is partly a human construction. This is because of the use of an imaginary unit expressible by the number i as a solution in the domain of real numbers of the unsolvable equation $x^2 = -1$. Solving such an equation requires the additional completion of numbers that are an artificial creation of man. This contradicts the rules of square roots in real numbers, which do not admit a square root of a negative number.

"In the early 1960s, Robinson succeeded in developing a non-standard analysis that yielded the so-called hyperreal numbers, i.e., a system within which there are both infinitely small and infinitely large numbers" (Kvasz, 2012: 109). I do not consider infinitesimally small and infinitely large numbers as numbers that exist realistically; they are constructions of man. Such a number simply cannot exist as a real number. Such numbers are useful; they are used in calculating limits, but they are not numbers outside the man-made world.

Maco postulates an argument that suggests that it makes no sense to consider two incompatible planes of mathematical objects, as I have argued in my study (Ambrozy, 2019). Maco seeks a positive assessment of the role of intuitionism in mathematics (Maco, 2016), to which I have no objection. "At the turn of the 18th and 19th centuries, the mathematicians recognised that their creations had not been formulated in a style of the deductive method of Euclid" (Bukovsky, 2011: 70). After all, heuristics within intuition are behind many discoveries and mathematical constructions. An example of a successful mathematician who worked with such a method can be found in Henri Poincaré.

I advocate the view postulated in Maco's study: "Humans create some mathematical objects; others are in turn discovered by humans" (Maco, 2015: 518). He ranks L. Kronecker as the first to hold the view. He is right, but I consider Kronecker's emphatic view that everything except integers is man's work to be extreme purism. According to him, C. F. Gauss is also in this line, admittedly not in close agreement with Kronecker's purism. I add that such a view was also held by the eminent Slovak topologist Lev Bukovský, with whom I have discussed the issue several times. Maco tries to argue in a Wittgensteinian way. Maco argues first with the history of how ancient mathematics got to irrational numbers. I agree with Maco that ancient Greek mathematics was limited to such expressions. He devotes a large part of the text to analysing Dirac's δ-function. As Maco writes, "It is simply, much easier (unencumbered by knowledge) to experience a newly presented object as a 'construct', which means nothing more than a 'creation' with connotations of arbitrariness" (Maco, 2015: 524). These were attempts considered unusual by the standards of contemporary mathematicians, and Dirac was aware of the problematic nature of this function, mentioned by J. von Neumann and referred to as eigenfunctions. I fully agree with Maco that the historical context has no bearing on whether we invent or construct mathematical objects, and this is a philosophical question, not a question of the history of mathematics or mathematics itself.

Maco claims that mathematicians have merely encountered problems. Of course, the history of mathematics does not change whether a mathematical object was discovered or created; that is irrelevant to historians of mathematics. Moreover, the fact that many mathematical objects exist independently of humans and can be constructed is certainly true. Here, we might be confused by the question that we are not concerned with whether an object was discovered or constructed at a given time and, in a particular case, in the history of science. The point is what its ontological status is, and thus whether it was in principle discoverable (even though it may have been constructed in the history of mathematics), or as Maco writes, whether we can discover them. They exist objectively, or whether they exist only for us and we can only construct them. This does not rule out that we have, by construction in some cases, arrived at an objectively existing mathematical object. After all, even positive integers can be constructed, as Russell and Whitehead showed in Principia Mathematica. Most of mathematics has been constructed in this way based on type theory. "The price we have to pay for this is disproportionately high: we have to squeeze the whole of mathematics into a considerably complex and artificial type structure and, as a precaution, exclude from it also a whole series of propositions and definitions which do not lead to controversy at all, but only transgress against the hierarchy of type theory" (Zlatoš, 1995: 109).

Mathematical objects that can only be constructed, cannot be discovered, and thus exist only for us are constructed in contradiction to existing rules. They are problematic; they introduce a problematic component into mathematics that does not conform to mathematical rules. Curved space can be intuitively imagined; even general relativity teaches us about its reality, and even so, it corresponds to empirical reality (which is not necessary for its objective existence). Complex numbers go against the rules of subtraction in the domain of real numbers, so they are a construct.

However, they are very useful entities from a pragmatic point of view, for example, in calculations related to the projection of the construction of a bridge.

According to Maco, Wittgenstein "constantly emphasises that in mathematics, we are not discoverers but inventors" (Maco, 2015: 527). This may be true, but it is certainly not true of Wittgenstein's early period. In the philosophy of mathematics, Wittgenstein changed his views considerably during his lifetime. Unlike Wittgenstein's general work, the philosophy of mathematics, which can be periodised triadically, has two basic stages - early and later. While in the first period of the philosophy of mathematics, Wittgenstein tended to the views he expressed in the Tractatus; his mathematical constructivism marked the second period. We do not consider Maco's interpretation of Wittgenstein's philosophy of mathematics when he abandoned the positions of the Tractatus to be relevant. We do mathematics, we create mathematics, thinks the Viennese philosopher. He sees it as pure syntax. Mathematics is a calculus that performs operations. We construct concrete statements according to the rules of calculus; mathematical constructivism is radically defended. Wittgenstein was suspicious of non-constructivist proofs. He regarded mathematics as a human practice. Wittgenstein criticizes many mathematical entities and practices; as we know, he rejects the validity of Gödel's theorems, he rejects mathematical induction, he thinks irrational numbers only make sense as rules, he rejects finitism in mathematics, etc. Thus, he expresses specific views within the philosophy of mathematics, namely in Remarks on the Foundations of Mathematics, Philosophical Investigations, and Philosophical Remarks. I do not believe that Wittgenstein's concrete positions on the philosophy of mathematics are "mere appearances" (Maco, 2015: 528).

The argument used to separate independently existing and constructed mathematical arguments is sufficiently Wittgensteinian. The introduction of a new mathematical object is considered to be purely constructivist insofar as it changes previously clear mathematical concepts and rules in the sense of contradicting them. "Logical belief, that is, the law of contradiction that we have held for millennia, is still valid in C-logic" (Hao, 2023: 28) (paraconsistent logic). According to Joaquin, the deepest paradox of deontic logic is simply ill-formulated (Joaquin, 2023). For example, the law of contradiction is also used in metaethics - cf. (Konstańczak, 2017). The bending of the space is just a qualitative enrichment, but introducing the square root of a negative number is clearly against the rules of arithmetic. It is a construct of man that is highly useful and applicable. However, Wittgenstein regards logical rules that cannot normally be rejected. Meanwhile, Wittgenstein sees correspondence and rule as related to the extent that, as one learns to use one concept, one also learns to use the other simultaneously. "According to Wittgenstein's account, our agreement in certain facts - or a certain form of life - is also responsible for how we habitually make judgments" (Čana, 2011: 114). Čana considers life form a key concept in understanding Wittgenstein's rules (Čana, 2016). Insofar as, as with Maco, "we can thereby establish some new 'language game'" (Maco, 2015: 528), it is a certain loose use of reasoning in a Wittgensteinian way.

Conclusion

The present paper attempts to justify dividing the ontological nature of mathematical objects into two groups. It builds on my earlier study (Ambrozy, 2019). I argue that one group of mathematical objects exists independently, independent of humans and their intervention. The other group of objects is a human construct. A typical example for the first group is the existence of natural numbers, and for the second group, the existence of complex numbers. I attempt to answer questions from the philosopher Maco's correspondence response to the nature of mathematical objects such as hyperreal numbers, Lie groups, Lie algebras, Hilbert space, and others. I have endeavoured to decompose the queried objects into various sub-components and answer the questions this way. While we consider hyperreal numbers and Hilbert space as human constructs, Lie groups and Lie algebras have a status depending on whether they are complex or not. To the objection that the group of human constructs includes negative numbers since, in the history of mathematics, mathematicians have treated them with reserve, I reply that this has not been the case throughout the world. According to such a procedure, we should also consider the higher positive integer natural number a construct since some natural peoples have no conceptual name for them; they do not know them. I also attempt to follow Wittgenstein's approach to mathematics, who also expressed particular views on the philosophy of mathematics, arguing that human mathematical constructions are based on violations of mathematical rules.

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